Warsaw University of Technology
Faculty of Power and Aeronautical Engineering
WARSAW UNIVERSITY OF TECHNOLOGY

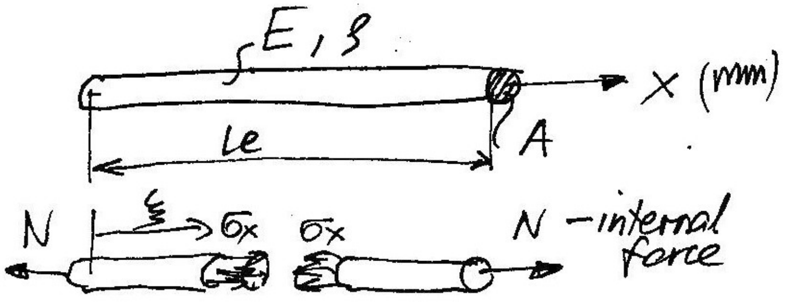
Institute of Aeronautics and Applied Mechanics

Finite element method (FEM)

Bar finite element

04.2021

A bar finite element (1D)

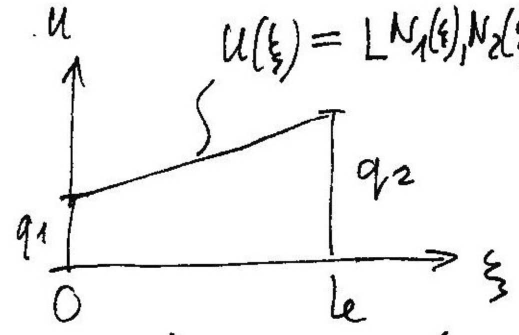


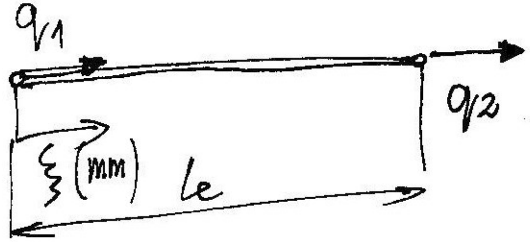
normal stress
constant in the cross-section A

ξ – local coordinate

$$d\xi = dx$$

$$u(\xi) = \lfloor N_1(\xi), N_2(\xi) \rfloor \cdot \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e$$





shape functions: $N_1(\xi) = 1 - \frac{\xi}{L_e}$

$N_2(\xi) = \frac{\xi}{L_e}$

$$\varepsilon_x = \frac{du}{dx} = \frac{du}{d\xi} \cdot \underbrace{\frac{d\xi}{dx}}_{1} = \frac{d\left(\lfloor N_1, N_2 \rfloor \cdot \{q\}_e\right)}{d\xi} =$$

$$= \frac{d\left(\lfloor N_1, N_2 \rfloor\right)}{d\xi} \cdot \underset{2\times1}{\{q\}_e} + \underbrace{\frac{d\underset{2\times1}{\{q\}_e}}{d\xi}}_{=0} \cdot \lfloor N_1, N_2 \rfloor = \left\lfloor \frac{dN_1}{d\xi}, \frac{dN_2}{d\xi} \right\rfloor \cdot \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e =$$

$$\left.\begin{aligned}
\varepsilon_x &= -\frac{1}{l_e}\cdot q_1 + \frac{1}{l_e}\cdot q_2 = \frac{q_2-q_1}{l_e} \\
\sigma_x &= E\,\varepsilon_x = \frac{E(q_2-q_1)}{l_e} \\
N &= \sigma_x \cdot A = \frac{EA}{l_e}(q_2-q_1)
\end{aligned}\right\} = \text{const in the finite element}$$

elastic strain energy:

$$U_e = \frac{1}{2}\int_{\Omega_e}\sigma_x\varepsilon_x\, d\Omega_e = \frac{1}{2}\int_0^{l_e}\sigma_x\varepsilon_x \int_A dA\, d\xi = \frac{EA}{2}\int_0^{l_e}\underset{\displaystyle \lfloor q\rfloor_e \cdot \begin{Bmatrix}\frac{dN_1}{d\xi}\\ \frac{dN_2}{d\xi}\end{Bmatrix}}{\underset{\parallel}{\frac{du}{d\xi}}}\cdot\underset{\displaystyle \left\lfloor \frac{dN_1}{d\xi}, \frac{dN_2}{d\xi}\right\rfloor\cdot\{q\}_e}{\underset{\parallel}{\frac{du}{d\xi}}}\, d\xi =$$

$$= \frac{1}{2}\underset{1\times 2}{\lfloor q\rfloor_e}\cdot EA\int_0^{l_e}\left[\begin{array}{c:c}\frac{dN_1}{d\xi}\frac{dN_1}{d\xi} & \frac{dN_1}{d\xi}\frac{dN_2}{d\xi}\\ \hdashline \frac{dN_2}{d\xi}\frac{dN_1}{d\xi} & \frac{dN_2}{d\xi}\frac{dN_2}{d\xi}\end{array}\right]d\xi\cdot\underset{2\times 1}{\{q\}_e} =$$

$$U_e = \frac{1}{2} \underset{1\times 2}{\lfloor q \rfloor_e} \cdot EA \left[\begin{array}{c|c} \int_0^{l_e} \left(-\frac{1}{l_e}\right)\left(-\frac{1}{l_e}\right) d\xi & \int_0^{l_e} \left(-\frac{1}{l_e}\right)\left(\frac{1}{l_e}\right) d\xi \\ \hline \int_0^{l_e} \left(\frac{1}{l_e}\right)\left(-\frac{1}{l_e}\right) d\xi & \int_0^{l_e} \left(\frac{1}{l_e}\right)\left(\frac{1}{l_e}\right) d\xi \end{array} \right] \cdot \{q\}_e =$$

$$= \frac{1}{2} \underset{1\times 2}{\lfloor q \rfloor_e} \underbrace{\frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{\text{local stiffness matrix } \underset{2\times 2}{[k]_e}} \cdot \{q\}_e = \frac{1}{2} \underset{1\times 2}{\lfloor q \rfloor_e} \underset{2\times 2}{[k]_e} \underset{2\times 1}{\{q\}_e}$$

potential energy of loading due to mass forces:

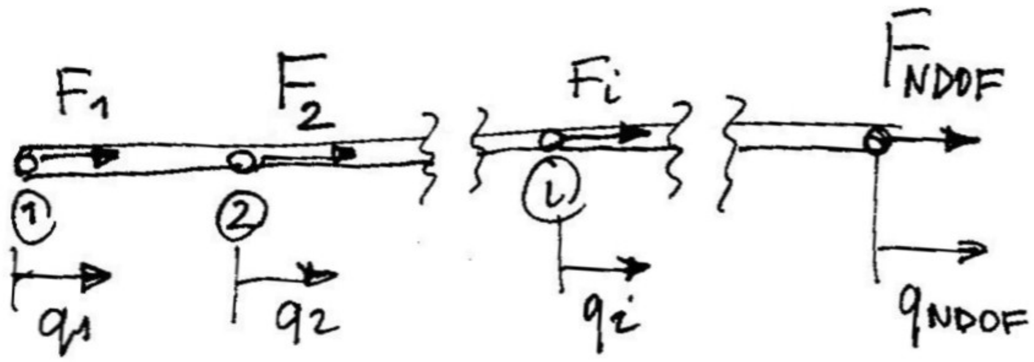
$$W_e = \int_{\Omega_e} \lfloor X \rfloor \cdot \{u\} \, d\Omega_e = \int_{\Omega_e} \rho \left(a_x \cdot u + \underset{0}{\underset{\parallel}{a_y}} \cdot v + \underset{0}{\underset{\parallel}{a_z}} \cdot w \right) d\Omega_e =$$

$$= \int_0^{l_e} \rho a_x \cdot u \int_A dA \, d\xi = \int_0^{l_e} \underbrace{\rho A a_x}_{\text{traction } p(\xi) \left(\frac{N}{m}\right)} \cdot u \, d\xi = \int_0^{l_e} p(\xi) u \, d\xi =$$

$$= \int_0^{l_e} p(\xi) \cdot \underset{1\times 2}{\lfloor N \rfloor} \cdot \underset{2\times 1}{\{q\}_e} \, d\xi = \lfloor \underbrace{\int_0^{l_e} p(\xi) \cdot N_1 \, d\xi}_{F_{1e}} , \underbrace{\int_0^{l_e} p(\xi) \cdot N_2 \, d\xi}_{F_{2e}} \rfloor \cdot \underset{2\times 1}{\{q\}_e} =$$

$$= \underset{\text{equivalent load vector}}{\lfloor F \rfloor_e} \cdot \{q\}_e$$

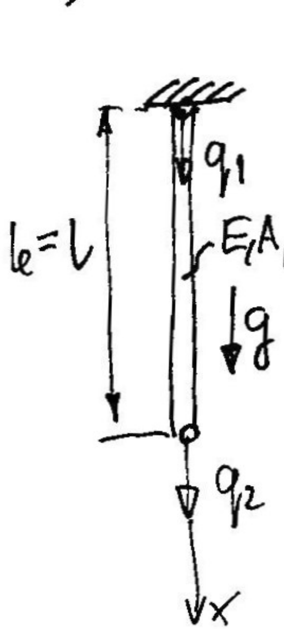
potential energy of nodal loads

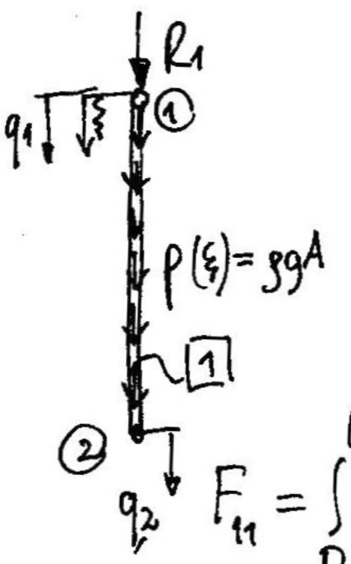


$$W^n = \sum_{i=1}^{NDOF} F_i q_i$$

Example. Find displacement, strain and stress in a finite element model of a bar loaded by gravity. Use one and two FEs. Compare a FE solution with an exact solution and the Ritz solution.

1°) one finite element:





$$\underset{2\times1}{\{q\}_1} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_1 , \quad \underset{2\times1}{\{q\}} = \{q\}_1$$

$$\underset{2\times2}{[k]_1} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \underset{2\times2}{[K]} = \underset{2\times2}{[k]_1^*}$$

$$\underset{1\times2}{\lfloor F \rfloor_1} = \lfloor F_{11}, \; F_{21} \rfloor$$

$$F_{11} = \int_0^L p N_1(\xi)\, d\xi = \rho g A \int_0^L \left(1 - \frac{\xi}{L}\right) d\xi = \rho g A \left(\xi - \frac{\xi^2}{2L}\right)\Bigg|_0^L = \frac{\rho g A L}{2}$$

$\xi = x$

$$F_{21} = \int_0^L p N_2(\xi)\, d\xi = \frac{\rho g A L}{2} , \quad \underset{1\times2}{\lfloor F \rfloor^e} = \underset{1\times2}{\lfloor F \rfloor_1} = \underset{1\times2}{\lfloor F \rfloor_1^*}$$

$$\underset{1\times2}{\lfloor F \rfloor^{n}} = \lfloor R_1, 0 \rfloor \quad , \quad \underset{1\times2}{\lfloor F \rfloor} = \underset{1\times2}{\lfloor F \rfloor^{e}} + \underset{1\times2}{\lfloor F \rfloor^{n}} = \left\lfloor \frac{\rho g A L}{2} + R_1 \,,\, \frac{\rho g A L}{2} \right\rfloor$$

$$\underset{2\times2}{[K]} \cdot \underset{2\times1}{\{q\}} = \underset{2\times1}{\{F\}} \quad \Rightarrow \quad \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} \frac{\rho g A L}{2} + R_1 \\ \frac{\rho g A L}{2} \end{Bmatrix}$$

+ boundary condition $q_1 = 0$

$$\frac{EA}{L} \cdot q_2 = \frac{\rho g A L}{2} \quad \Rightarrow \quad q_2 = \frac{\rho g L^2}{2E}$$

displacement in the element:

$$u(\xi) = u(x) = \lfloor N \rfloor \cdot \{q\}_1 = N_1 \cdot q_1 + N_2 \cdot q_2 = \frac{\xi}{L}\frac{\rho g L^2}{2E} = \frac{\rho g L}{2E} \cdot \xi$$

strain in the element:

$$\varepsilon_x = \frac{q_2 - q_1}{L} = \frac{\rho g L}{2E} = \text{const.}$$

stress in the element:

$$\sigma_x = E\,\varepsilon_x = \frac{\rho g L}{2} = \text{const.}$$

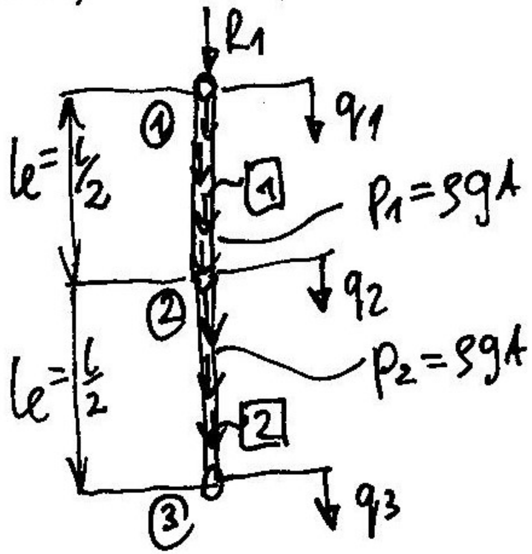
reaction:

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ \frac{\rho g L^2}{2E} \end{Bmatrix} = \begin{Bmatrix} \frac{\rho g A L}{2} + R_1 \\ \frac{\rho g A L}{2} \end{Bmatrix}$$

$$\frac{EA}{L}\left(1 \cdot 0 - 1 \cdot \frac{\rho g L^2}{2E}\right) = \frac{\rho g A L}{2} + R_1$$

$$R_1 = -\frac{\rho g A L}{2} - \frac{\rho g A L}{2} = -\rho g A L = -m \cdot g$$

2^c) two finite elements:



$$\underset{2\times 1}{\{q\}_1} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_1 \qquad \underset{3\times 1}{\{q\}} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

$$\underset{2\times 1}{\{q\}_2} = \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}_2$$

$$\underset{2\times 2}{[k]_1} = \frac{2EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \; ; \; \underset{3\times 3}{[k]_1^*} = \frac{2EA}{l}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underset{2\times 2}{[k]_2} = \frac{2EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \; ; \; \underset{3\times 3}{[k]_2^*} = \frac{2EA}{l}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\underset{3\times 3}{[K]} = \underset{3\times 3}{[k]_1^*} + \underset{3\times 3}{[k]_2^*} = \frac{2EA}{l}\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}}_{\det \underset{3\times 3}{[K]} = 0}$$

$$\underset{1\times 2}{\lfloor F \rfloor}_1 = \left\lfloor \int_0^{\frac{L}{2}} p_1 N_1 d\xi , \int_0^{\frac{L}{2}} p_1 N_2 d\xi \right\rfloor = \frac{\rho g A L}{4} \lfloor 1, 1 \rfloor$$

$$\underset{1\times 2}{\lfloor F \rfloor}_2 = \left\lfloor \int_0^{\frac{L}{2}} p_2 N_1 d\xi , \int_0^{\frac{L}{2}} p_2 N_2 d\xi \right\rfloor = \frac{\rho g A L}{4} \lfloor 1, 1 \rfloor$$

$$; \quad \underset{1\times 3}{\lfloor F \rfloor}_1^* = \frac{\rho g A L}{4} \lfloor 1, 1, 0 \rfloor$$

$$; \quad \underset{1\times 3}{\lfloor F \rfloor}_2^* = \frac{\rho g A L}{4} \lfloor 0, 1, 1 \rfloor$$

$$\underset{1\times 3}{\lfloor F \rfloor}^e = \sum_{e=1}^{2} \underset{1\times 3}{\lfloor F \rfloor}_e^* = \underset{1\times 3}{\lfloor F \rfloor}_1^* + \underset{1\times 3}{\lfloor F \rfloor}_2^* = \frac{\rho g A L}{4} \lfloor 1, 2, 1 \rfloor$$

$$\underset{1\times 3}{\lfloor F \rfloor}^n = \lfloor R_1, 0, 0 \rfloor \quad , \quad \underset{1\times 3}{\lfloor F \rfloor} = \left\lfloor R_1 + \frac{\rho g A L}{4}, \frac{\rho g A L}{2}, \frac{\rho g A L}{4} \right\rfloor$$

$$\underset{3\times 3}{[K]} \cdot \underset{3\times 1}{\{q\}} = \underset{3\times 1}{\{F\}} \qquad \frac{2EA}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} R_1 + \frac{\rho g A L}{4} \\ \frac{\rho g A L}{2} \\ \frac{\rho g A L}{4} \end{Bmatrix}$$

+ boundary condition $q_1 = 0$

$$\underbrace{\frac{2EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}_{\underset{2\times 2}{[K]}} \cdot \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} \rho g \frac{AL}{2} \\ \rho g \frac{AL}{4} \end{Bmatrix}$$

$$\det \underset{2\times 2}{[K]} = \left(\frac{2EA}{L}\right)^2 \cdot \left(2\cdot 1 - (-1)\cdot(-1)\right) = \left(\frac{2EA}{L}\right)^2$$

$$\underset{2\times 2}{[K^c]} = \frac{2EA}{L}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \underset{2\times 2}{[K^c]}^T$$

$$[K]^{-1} = \frac{1}{\det \underset{2\times 2}{[K]}} \cdot \underset{2\times 2}{[K^c]}^T = \frac{L}{2EA}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = [K]^{-1} \cdot \frac{\rho g A L}{4}\begin{Bmatrix} 2 \\ 1 \end{Bmatrix} = \frac{\rho g A L^2}{8EA}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} = \frac{\rho g L^2}{8E}\begin{Bmatrix} 3 \\ 4 \end{Bmatrix}$$

$$q_2 = \frac{3}{8}\frac{\rho g L^2}{E} \quad , \quad q_3 = \frac{\rho g L^2}{2E}$$

displacements in elements : FE solution in elements :

$\boxed{1}$: $u_1(\xi) = \underset{1\times 2}{\lfloor N \rfloor} \cdot \underset{2\times 1}{\{q\}_1} = \left(1 - \frac{\xi}{\left(\frac{L}{2}\right)}\right) \cdot q_1 + \frac{\xi}{\left(\frac{L}{2}\right)} \cdot q_2 =$

$$= \frac{2\xi}{L} \cdot \frac{3}{8} \frac{\varrho g L^2}{E} = \frac{3 \varrho g L}{4E} \cdot \xi$$

$x \in \langle 0, \frac{L}{2} \rangle$; $\xi = x$ $\Rightarrow$ $u(x) = \frac{3\varrho g L}{4E} \cdot x$

$u(0) = 0$, $u\left(\frac{L}{2}\right) = \frac{3}{8} \frac{\varrho g L^2}{E}$

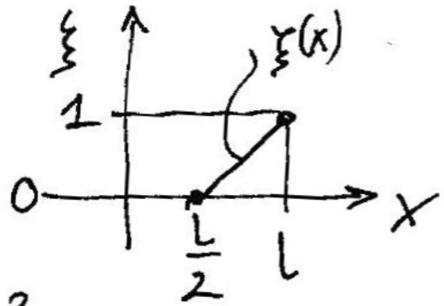
$$\varepsilon_{x_1} = \left\lfloor \frac{dN_1}{d\xi}, \frac{dN_2}{d\xi} \right\rfloor \cdot \underset{2\times 1}{\{q\}_1} = -\frac{2}{L} \cdot q_1 + \frac{2}{L} \cdot q_2 = \frac{3 \varrho g L}{4E}$$

$$\sigma_{x_1} = E \varepsilon_{x_1} = \frac{3 \varrho g L}{4}$$

$\boxed{2}$ $u_2(\xi) = \underset{1\times 2}{\lfloor N \rfloor} \cdot \underset{2\times 1}{\{q\}_2} = \left(1 - \frac{2\xi}{L}\right) q_2 + \frac{2\xi}{L}\cdot q_3 =$

$$= \frac{3}{8}\frac{\rho g L^2}{E} - \frac{3\rho g L}{4E}\cdot\xi + \frac{\rho g L}{E}\cdot\xi = \frac{\rho g L}{4E}\cdot\xi + \frac{3\rho g L^2}{8E}$$

$x \in \langle \frac{L}{2}, L \rangle$; $\xi(x) = x - \frac{L}{2}$



$$u_2(x) = \frac{\rho g L}{4E}\cdot x + \frac{\rho g L^2}{4E} = \frac{\rho g}{4E}\left(x\cdot L + L^2\right)$$

$u\left(\frac{L}{2}\right) = \frac{3}{8}\frac{\rho g L^2}{E}$, $u(L) = \frac{\rho g L^2}{2E}$

$$\varepsilon_{x_2} = \left\lfloor \frac{dN_1}{d\xi}, \frac{dN_2}{d\xi} \right\rfloor \cdot \{q\}_2 = -\frac{2}{L} q_2 + \frac{2}{L} q_3 = \frac{2(q_3 - q_2)}{L} = \frac{\rho g L}{4E}$$

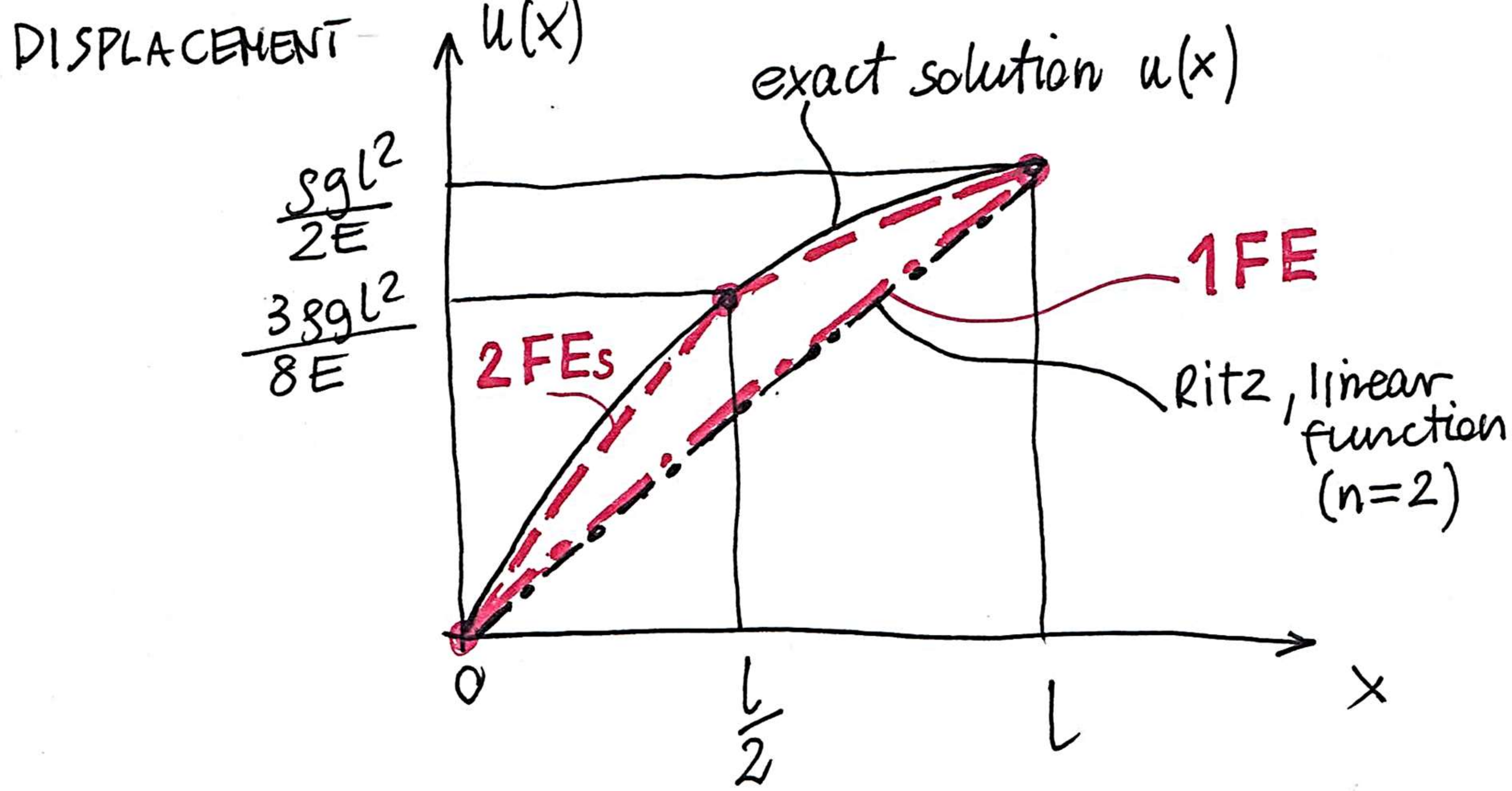
$$\sigma_{x_2} = E\,\varepsilon_{x_2} = \frac{\rho g L}{4}$$

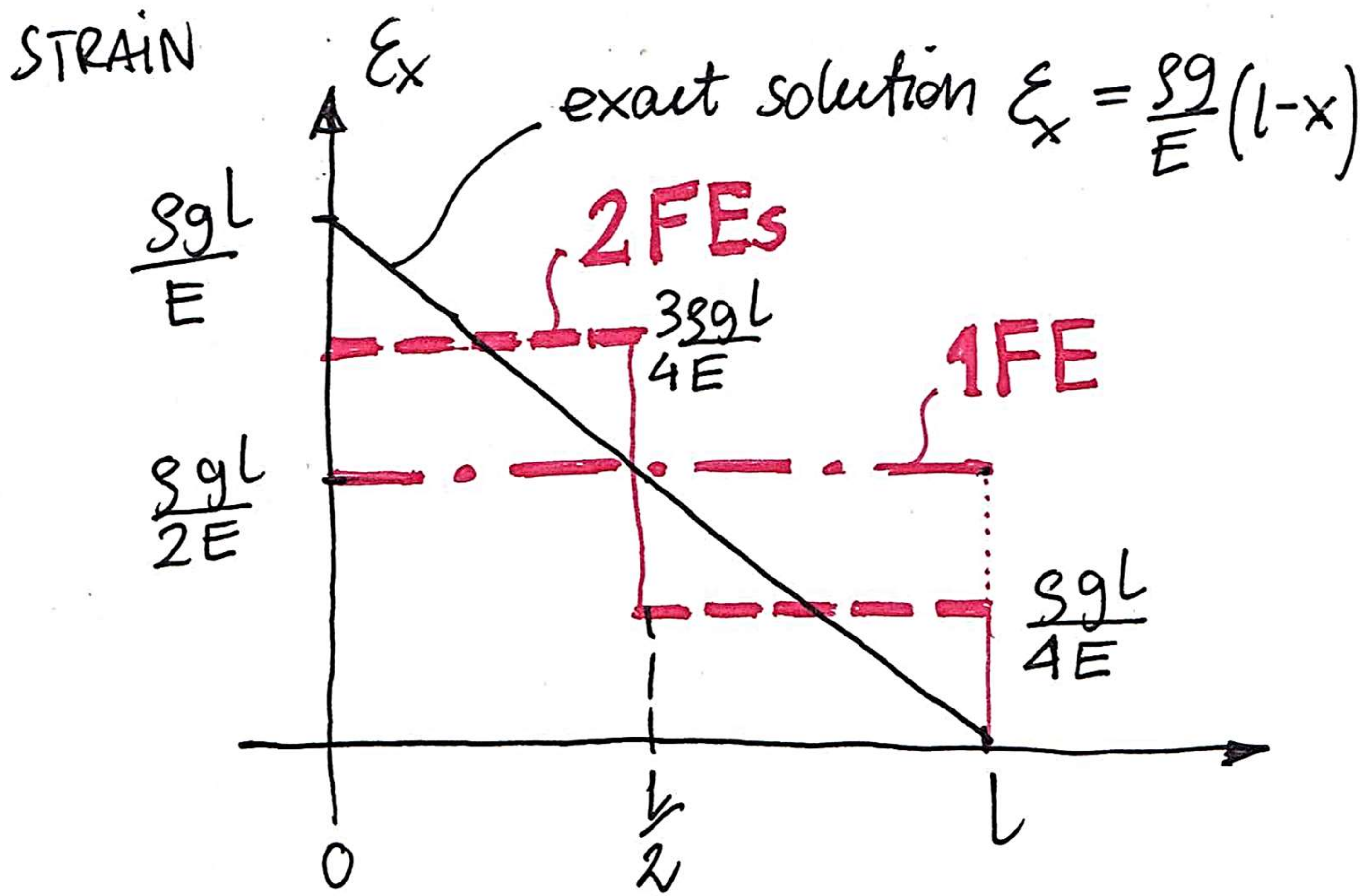
reaction : $\frac{2EA}{l}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\cdot\frac{\varrho g l^2}{8E}\begin{Bmatrix}0\\ 3\\ 4\end{Bmatrix}=\begin{Bmatrix}R_1+\frac{\varrho g A L}{4}\\ \vdots\\ \cdot\end{Bmatrix}$

$$\frac{2EA}{l}\left(1\cdot 0-1\cdot 3+0\cdot 4\right)\cdot\frac{\varrho g l^2}{8E}=R_1+\varrho g\frac{AL}{4}$$

$$-\frac{3}{4}\varrho g A L-\frac{1}{4}\varrho g A L=R_1$$

$$R_1=-\varrho g A L=-mg$$


DISPLACEMENT
u(x)
exact solution u(x)
ρgl²/2E
3ρgl²/8E
1FE
2FEs
Ritz, linear function (n=2)
0
l/2
l
x



STRAIN
ε_x
exact solution $\varepsilon_x = \frac{\rho g}{E}(l-x)$
$\frac{\rho g l}{E}$
2FEs
$\frac{3\rho g l}{4E}$
1FE
$\frac{\rho g l}{2E}$
$\frac{\rho g l}{4E}$
0
$l/2$
l

10 finite elements